ISOLDE X

Sevilla and Islantilla, 2 – 8 June 2005

A branch-and-price framework for single-source capacitated location problems

A. Ceselli, F. Liberatore, G. Righini

Università degli Studi di Milano

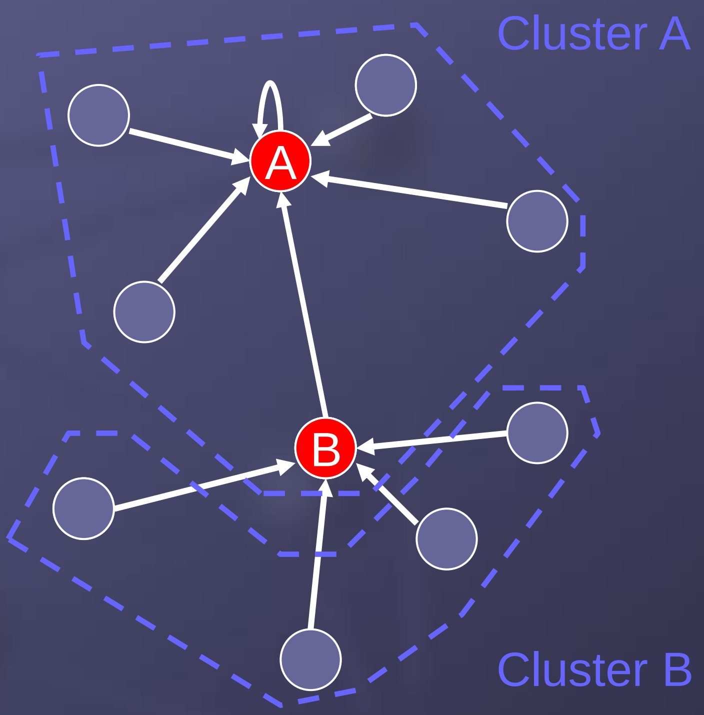
Dipartimento di Tecnologie dell'Informazione

Outline of the talk

1. An overview of single-source capacitated location models
2. Regional constraints
3. A branch and price framework
    1. Formulations
    2. Main algorithm
4. Experimental analysis

Single Source Capacitated Facility Location Problem (SS-CFLP)

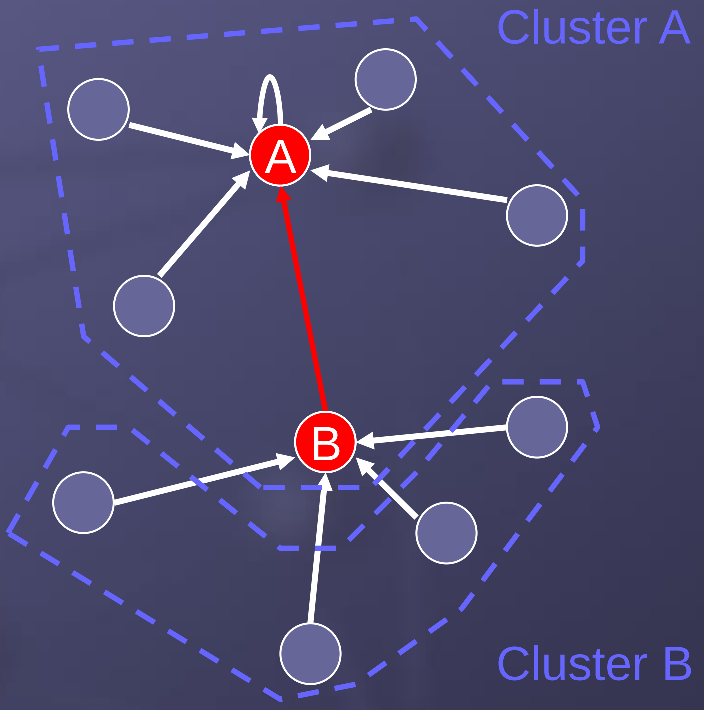
- Data:
  - A set of users (I), each with
    - a demand w_i
  - A set of location sites (J), each with
    - a potential production capacity Q_j
    - a setup cost f_j
  - A distance matrix (d_{ij}) between users and location sites
- Variables:
  - Locate a set of facilities (y_j) and assign users to facilities (x_{ij})
- Constraints:
  - Capacity constraints
  - Single source constraints
  - (Cardinality constraints …)



- Objective: minimize (average) distance between users and facilities

Capacitated Concentrator Location Problem (CCLP)

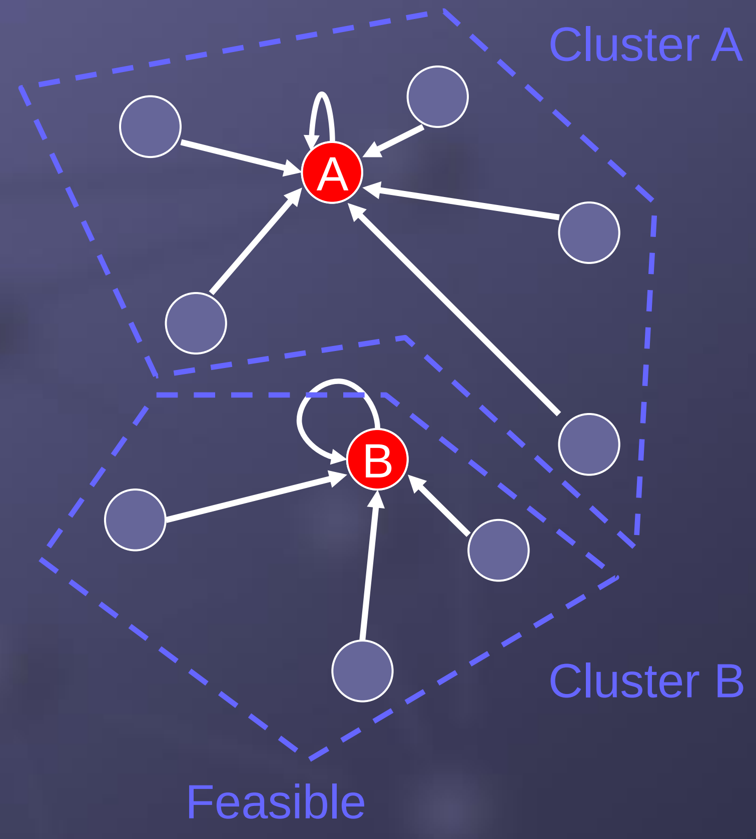
- Common in Network design problems
- The set of user locations and potential production sites coincide (I = J)
- Facilities are called “concentrators”
- Each concentrator must belong to its cluster



B does not belong to its cluster

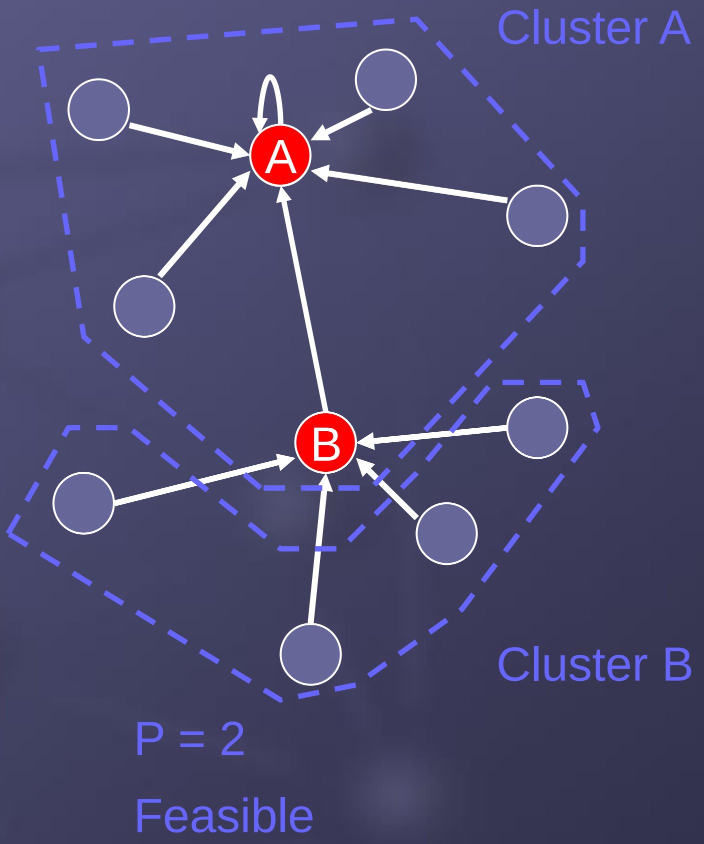
Capacitated Concentrator Location Problem (CCLP)

- Common in Network design problems
- The set of user locations and potential production sites coincide (I = J)
- Facilities are called "concentrators"
- Each concentrator must belong to its cluster



Capacitated P-Median Problem (CPMP)

- Facilities are called “medians”
- No setup costs
- The number of facilities to be activated is given (P)

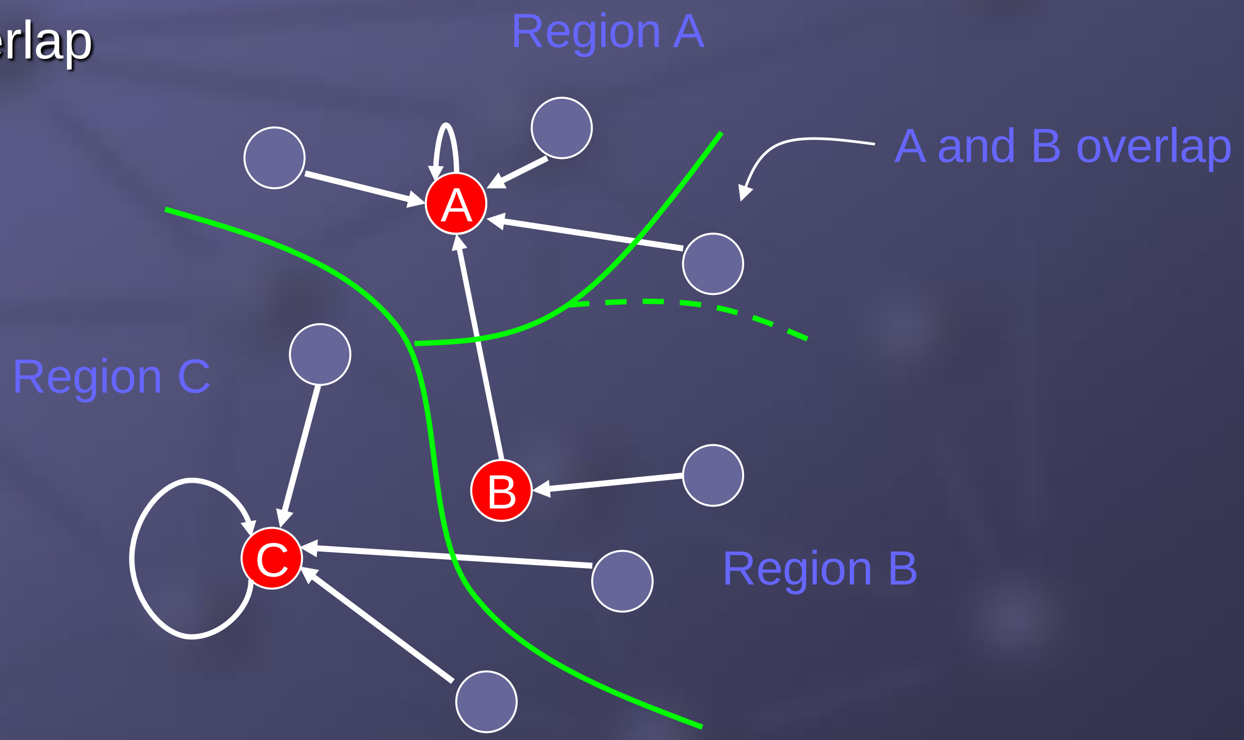


Compact formulations

		SS-CFLP	CCLP	CPMP	CPCLP
$\min z = \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{j \in J} f_j y_j$		X	X		
s.t.					
$\sum_{j \in J} x_{ij} = 1$	$\forall i \in I$	X	X	X	X
$\sum_{i \in I} w_i x_{ij} \le Q_j y_j$	$\forall j \in J$	X	X	X	X
$\sum_{j \in J} y_j \le p$				X	X
$y_j = x_{jj}$	$\forall j \in J$		X		X
$x_{ij}, y_j \in \{0,1\}$	$\forall i \in I, \forall j \in J$	X	X	X	X

Regional constraints

- Subsets of J represent regions
- Lower/Upper bounds on the number of facilites in each region
- Regions could overlap



Regionally constrained location problems

$$\min z = \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{j \in J} f_j y_j$$

$$s.t.$$

$$\begin{cases} \sum_{j \in J} x_{ij} = 1 & \forall i \in I \\ \sum_{i \in I} w_i x_{ij} \le Q_j y_j & \forall j \in J \\ \sum_{j \in R} y_j \le u_R & \forall R \in \boldsymbol{R} \\ \sum_{j \in R} y_j \ge l_R & \forall R \in \boldsymbol{R} \\ x_{ij}, y_j \in \{0,1\} & \forall i \in I, \forall j \in J \end{cases}$$

Reformulation

$$\min z = \sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij} + \sum_{j\in J} f_j y_j$$

s.t.

$$\begin{cases} \sum_{j\in J} x_{ij} = 1 \\ \sum_{i\in I} w_i x_{ij} \le Q_j y_j \\ \sum_{j\in R} y_j \le u_R \\ \sum_{j\in R} y_j \ge l_R \\ x_{ij}, y_j \in \{0,1\} \end{cases}$$

$$\min v = \sum_{j\in J}\sum_{k\in Z^j} (f_j + \sum_{i\in I} d_{ij}x_i^k) z_j^k$$

s.t.

$$\begin{cases} \sum_{j\in J}\sum_{k\in Z^j} x_i^k z_j^k = 1 & \forall i \in I \\ \sum_{k\in Z^j} z_j^k \le 1 & \forall j \in J \\ \sum_{j\in R}\sum_{k\in Z^j} z_j^k \le u_R & \forall R \in \boldsymbol{R} \\ \sum_{j\in R}\sum_{k\in Z^j} z_j^k \ge l_R & \forall R \in \boldsymbol{R} \\ z_j^k \in \{0,1\} & \forall j \in J, \forall k \in Z^j \end{cases}$$

$$c_j^k = f_j y_j^k + \sum_{i\in I} d_{ij} x_i^k \quad | \quad \sum_{i\in I} w_i x_i^k \le Q_j y_j^k, \quad x_i^k, y_j^k \in \{0,1\}$$

SS-CFLP

$$\min v = \sum_{j\in J}\sum_{k\in Z^j}(f_j + \sum_{i\in I} d_{ij}x_i^k)z_j^k$$

$s.t.$

$$\begin{cases} \sum_{j\in J}\sum_{k\in Z^j} x_i^k z_j^k = 1 \quad \geq & \forall i \in I \\ \sum_{k\in Z^j} z_j^k \leq 1 & \forall j \in J \\ \sum_{j\in R}\sum_{k\in Z^j} z_j^k \leq u_R & \forall R \in \mathbf{R} \\ \sum_{j\in R}\sum_{k\in Z^j} z_j^k \geq l_R & \forall R \in \mathbf{R} \\ z_j^k \in \{0,1\} & \forall j \in J, \forall k \in Z^j \end{cases}$$

$$c_j^k = f_j y_j^k + \sum_{i\in I} d_{ij}x_i^k \;\Big|\; \sum_{i\in I} w_i x_i^k \leq Q_j y_j^k, \quad x_i^k, y_j^k \in \{0,1\}$$

CCLP

$$\min v = \sum_{j\in J}\sum_{k\in Z^j}\left(f_j + \sum_{i\in I} d_{ij}x_i^k\right)z_j^k$$

$s.t.$

$$\begin{cases} \sum_{j\in J}\sum_{k\in Z^j} x_i^k z_j^k = 1 & \forall i \in I \\ \sum_{k\in Z^j} z_j^k \le 1 & \forall j \in J \\ \sum_{j\in R}\sum_{k\in Z^j} z_j^k \le u_R & \forall R \in \boldsymbol{R} \\ \sum_{j\in R}\sum_{k\in Z^j} z_j^k \ge l_R & \forall R \in \boldsymbol{R} \\ z_j^k \in \{0,1\} & \forall j \in J, \forall k \in Z^j \end{cases}$$

$$x_j^k$$

$$c_j^k = f_j y_j^k + \sum_{i\in I} d_{ij}x_i^k \;\middle|\; \sum_{i\in I} w_i x_i^k \le Q_j y_j^k, \quad x_i^k, y_j^k \in \{0,1\}$$

CPMP

$$\min v = \sum_{j\in J}\sum_{k\in Z^j}\left(f_j + \sum_{i\in I} d_{ij}x_i^k\right)z_j^k$$

$s.t.$

$$\begin{cases} \displaystyle\sum_{j\in J}\sum_{k\in Z^j} x_i^k z_j^k \overset{\geq}{=} 1 & \forall i \in I \\ \displaystyle\sum_{k\in Z^j} z_j^k \leq 1 & \forall j \in J \\ \displaystyle\sum_{j\in R}\sum_{k\in Z^j} z_j^k \leq u_R & \forall R \in \boldsymbol{R} \\ \displaystyle\sum_{j\in R}\sum_{k\in Z^j} z_j^k \geq l_R & \forall R \in \boldsymbol{R} \\ z_j^k \in \{0,1\} & \forall j \in J, \forall k \in Z^j \end{cases}$$

P

$|\boldsymbol{R}| = 1, R = J$

$$c_j^k = f_j y_j^k + \sum_{i\in I} d_{ij}x_i^k \;\middle|\; \sum_{i\in I} w_i x_i^k \leq Q_j y_j^k, \quad x_i^k, y_j^k \in \{0,1\}$$

LP Relaxation

$$\min v = \sum_{j\in J}\sum_{k\in Z^j}(f_j + \sum_{i\in I} d_{ij}x_i^k)z_j^k$$

$$s.t.$$

$$\begin{cases} \sum_{j\in J}\sum_{k\in Z^j} x_i^k z_j^k = 1 & \forall i \in I \\ \sum_{k\in Z^j} z_j^k \le 1 & \forall j \in J \\ \sum_{j\in R}\sum_{k\in Z^j} z_j^k \le u_R & \forall R \in \boldsymbol{R} \\ \sum_{j\in R}\sum_{k\in Z^j} z_j^k \ge l_R & \forall R \in \boldsymbol{R} \\ z_j^k \in \{0,1\} & \forall j \in J, \forall k \in Z^j \end{cases}$$

$$\min v = \sum_{j\in J}\sum_{k\in Z^j}(f_j + \sum_{i\in I} d_{ij}x_i^k)z_j^k$$

$$s.t.$$

$$\begin{cases} \sum_{j\in J}\sum_{k\in Z^j} x_i^k z_j^k \ge 1 & (\lambda_i) & \forall i \in I \\ -\sum_{k\in Z^j} z_j^k \ge -1 & (\mu_j) & \forall j \in J \\ -\sum_{j\in R}\sum_{k\in Z^j} z_j^k \ge -u_R & (\gamma_R^u) & \forall R \in \boldsymbol{R} \\ \sum_{j\in R}\sum_{k\in Z^j} z_j^k \ge l_R & (\gamma_R^l) & \forall R \in \boldsymbol{R} \\ 0 \le z_j^k (\le 1) & & \forall j \in J, \forall k \in Z^j \end{cases}$$

Pricing problem

$$\min \ \bar{c}_j^k = \sum_{i \in I} d_{ij} x_i^k - \sum_{i \in I} \lambda_i x_i^k + f_j + \mu_j - \sum_{R \in \mathcal{R} | j \in R} (\gamma_R^l - \gamma_R^u)$$

$$\begin{cases} \sum_{i \in I} w_i x_i^k \le Q_j \\ x_i^k \in \{0,1\} \quad \forall i \in I \end{cases}$$

$$\bar{c}_j^k = -\tau_j + \eta_j$$

$$\max \tau_j = \sum_{i \in I} (\lambda_i - d_{ij}) x_i^k$$

$s.t.$

$$\begin{cases} \sum_{i \in I} w_i x_i^k \le Q_j \\ x_i^k \in \{0,1\} \quad \forall i \in I \end{cases}$$

0-1 Knapsack Problem (NP-Hard)

Pisinger's MINKNAP Algorithm

$$\begin{array}{cl} & \\ \left(\begin{array}{c} x_1^k \\ \vdots \\ x_N^k \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{array}\right) & \begin{array}{l} \\ \\ \\ \\ \\ \\ \leftarrow j \\ \\ \\ \leftarrow R \\ \\ \end{array} \end{array}$$

Branching

- Rebuild a fractional solution for the original formulation:

$$\tilde{x}_{ij} = \sum_{k \in Z^j} x_i^k z_j^k$$

- Let M_i be the set of medians j to which i is assigned in a fractional solution ($\tilde{x}_{ij} \geq 0$)
- Let R_i be the set of medians j to which i is not assigned in a fractional solution ($\tilde{x}_{ij} = 0$)

Branching:

- GUB-constraints branching
- 1 – Constraint selection
  - Select i* with highest $|M_i|$

$$\sum_{j \in M_{i*} \cup R_{i*}} x_{i*j} = 1$$

- 2 – Partition
  - Partition M_{i*} in M_{i*}^- and M_{i*}^+
  - Partition R_{i*} in R_{i*}^- and R_{i*}^+
- 3 – Branch
  - Set, in one branch, $\sum_{j \in M_{i*}^- \cup R_{i*}^-} x_{i*j} = 0$
  - Set, in the other branch, $\sum_{j \in M_{i*}^+ \cup R_{i*}^+} x_{i*j} = 0$
- 4 – Search
  - Best bound first strategy

Primal Bound

- Coefficients:

$$\tilde{x}_{ij} = \sum_{k \in Z} x_i^k z_j^k$$ "Desirability" of assigning vertex i to facility j

$$\psi_j = \sum_{i \in I} \tilde{x}_{ij}$$ "Desirability" of locating a facility in j

- Location problem:
  - Select the most desirable sites, that contribute to satisfy region lower bounds
  - Select the most desirable sites, that do not violate region upper bounds
- Direct assignment (MTH-Like greedy allocation)
- Assignment through exchanges
  (Local search for feasibility)
- Solution improvement (shifts and swaps)

Lagrangean Relaxation

$$\min z = \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{j \in J} f_j y_j$$

s.t.

$$\begin{cases} \sum_{j \in J} x_{ij} = 1 \\ \sum_{i \in I} w_i x_{ij} \le Q_j y_j \\ \left.\begin{array}{l} \sum_{j \in R} y_j \le u_R \\ \sum_{j \in R} y_j \ge l_R \end{array}\right\} \\ x_{ij}, y_j \in \{0,1\} \end{cases}$$

$$\min z = \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{j \in J} f_j y_j +$$

$$- \sum_{j \in J} \lambda_j \left(\sum_{j \in J} x_{ij} - 1 \right) +$$

$$- \sum_{R \in \mathrm{R}} \gamma_R^u \left(u_R - \sum_{j \in R} y_j \right) - \sum_{R \in \mathrm{R}} \gamma_R^u \left(\sum_{j \in R} y_j - l_R \right)$$

s.t.

$$\begin{cases} \sum_{i \in I} w_i x_{ij} \le Q_j y_j \\ x_{ij}, y_j \in \{0,1\} \end{cases}$$

Lagrangean relaxation

- Valid dual bounds at each CG iteration
- Multiple pricing
  - (combined with subgradient optimization)
- Variable fixing
- Lagrangean heuristics

Experimental results

- C++
- ILOG CPLEX 8.1
- P IV 1.6 GHz
- Linux RedHat 9

- Dataset 1: Holmberg (71 istances) and Diaz-Fernandez (57 instances)
- Dataset 2: OR Library CPMP instances (20 instances)
- (Dataset 3: OR Library large-size PMP instances)

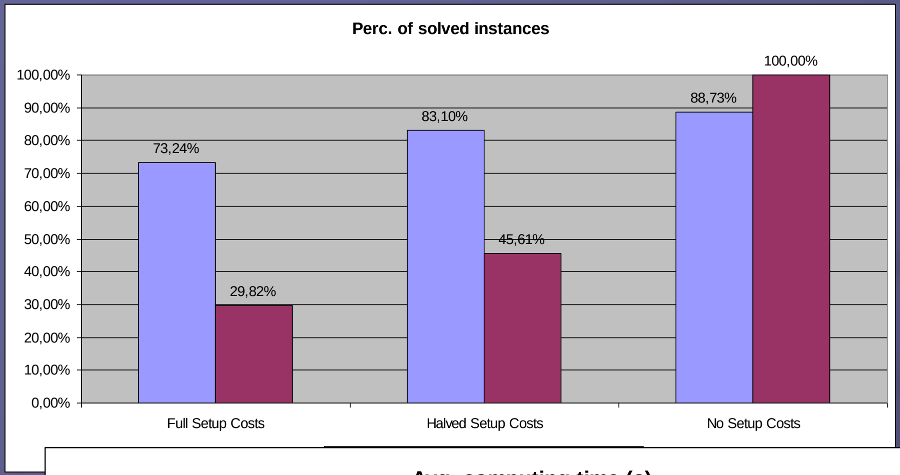
- Resource limitations:
  - Time limit: 60 minutes
  - Memory limit: 512 MB

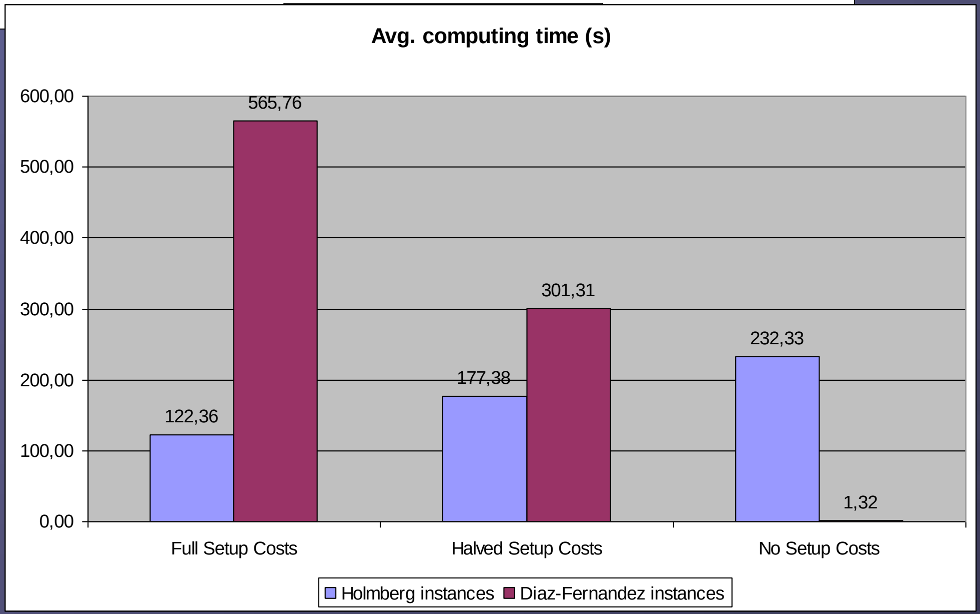
Experimental analysis

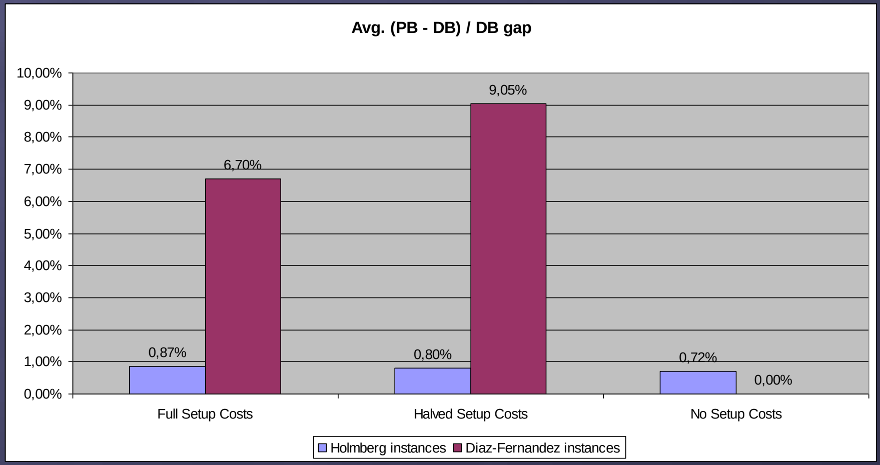
1. Cardinality constraints and setup costs
   1. Cardinality constraints in SS-CFLP problems
   2. Uniform VS non uniform setup costs
   3. CPMP, SS-CFLP and mixed models
2. Regional constraints
3. Concentrator models

Cardinality constraint in SS-CFLP

Dataset 1 + cardinality constraints (load factor = 0.8)





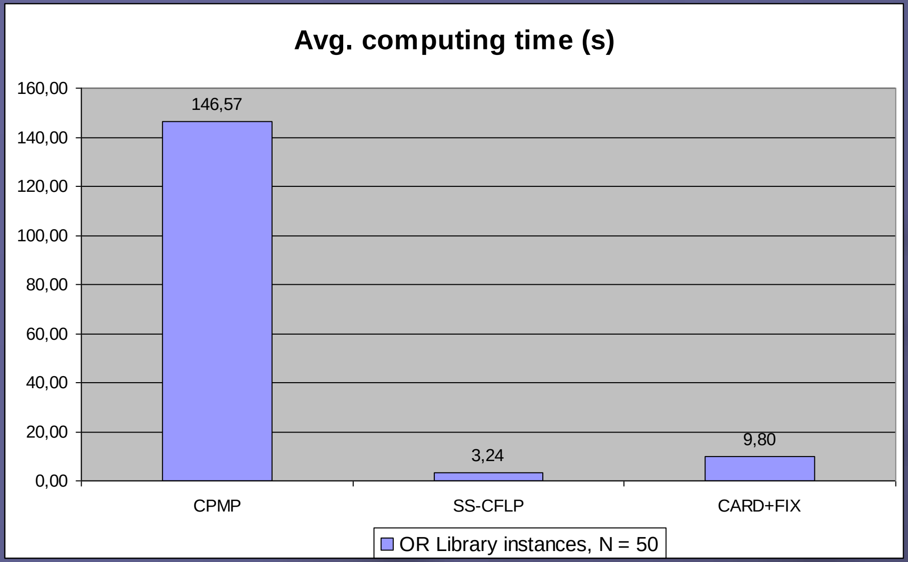


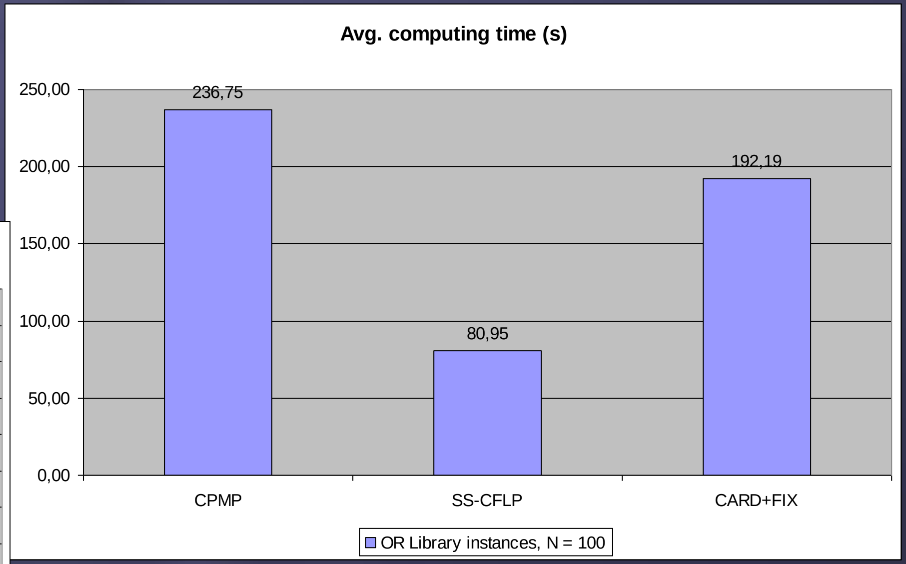
Uniform VS non uniform setup costs

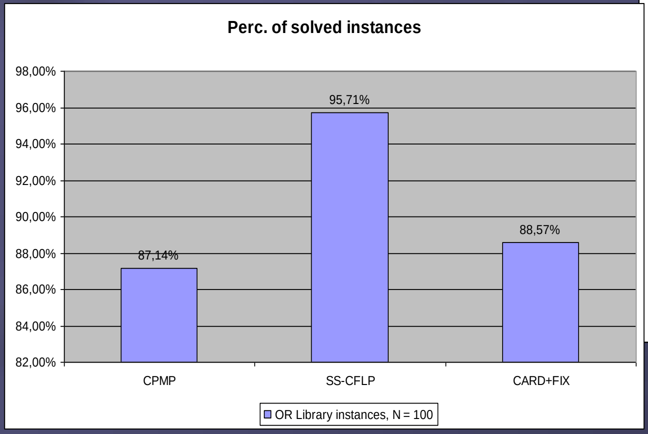
Diaz-Fernandez instances

	Non uniform	Uniform
Avg computing time (s)	565,76	156,36
Solved instances (57)	17	53
Avg. (PB - DB) / DB gap	6,70%	2,14%

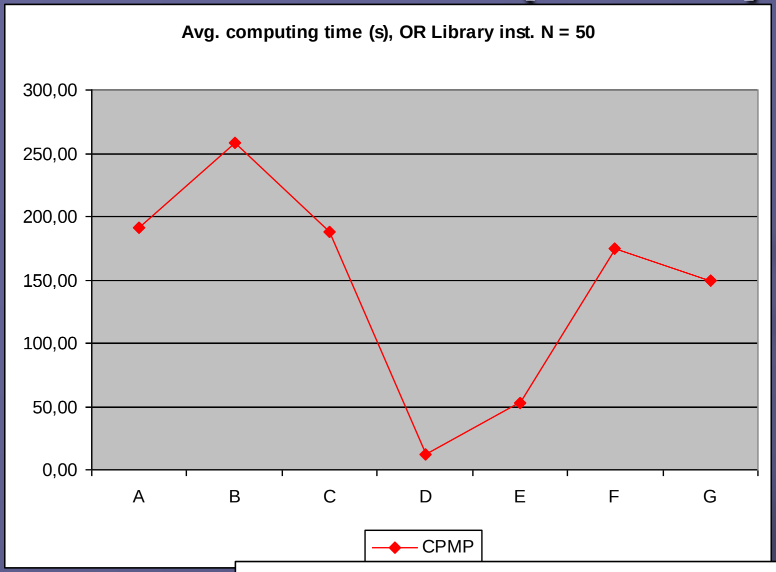
CPMP, SS-CFLP and mixed models

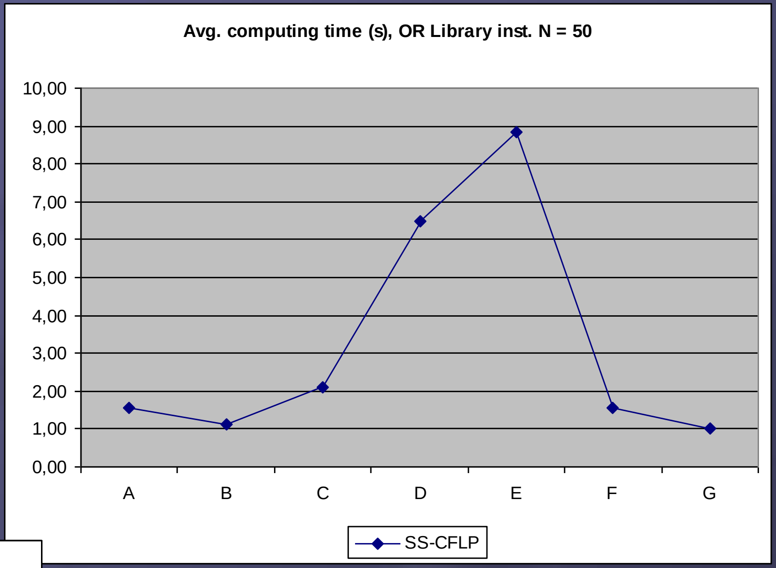


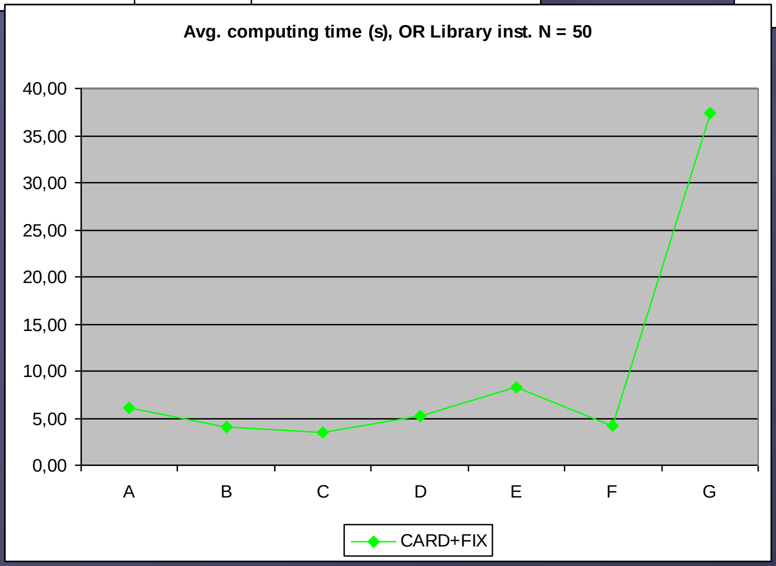




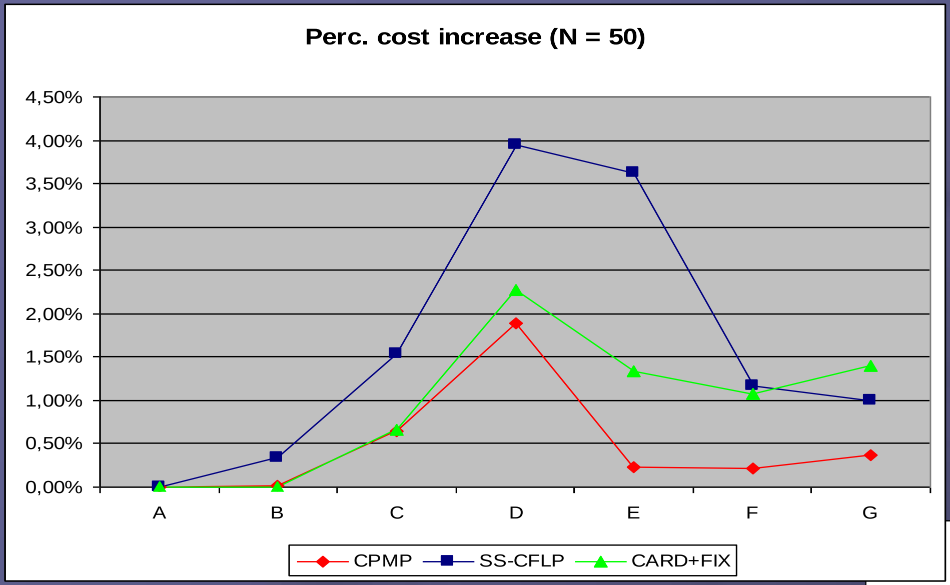
Regional constraints, N = 50 (computing time)

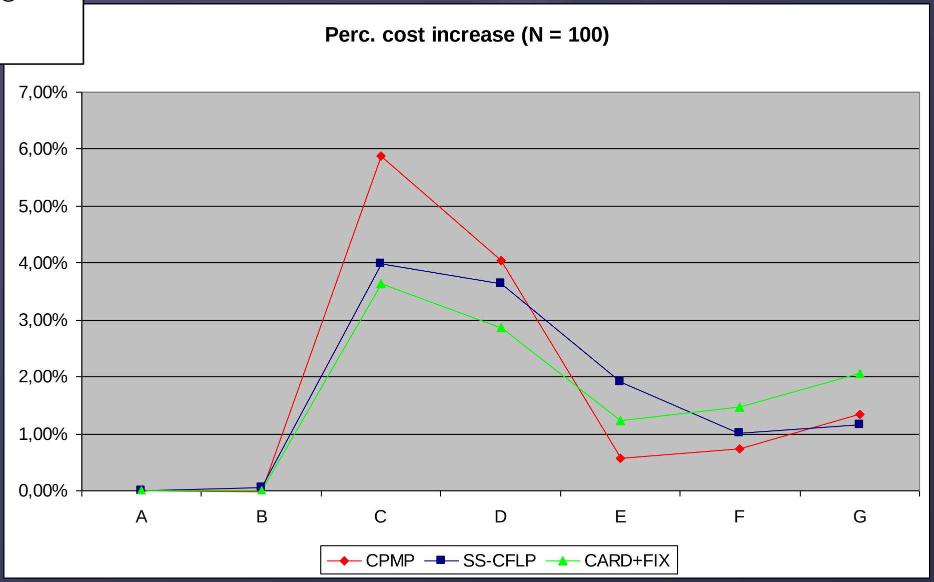




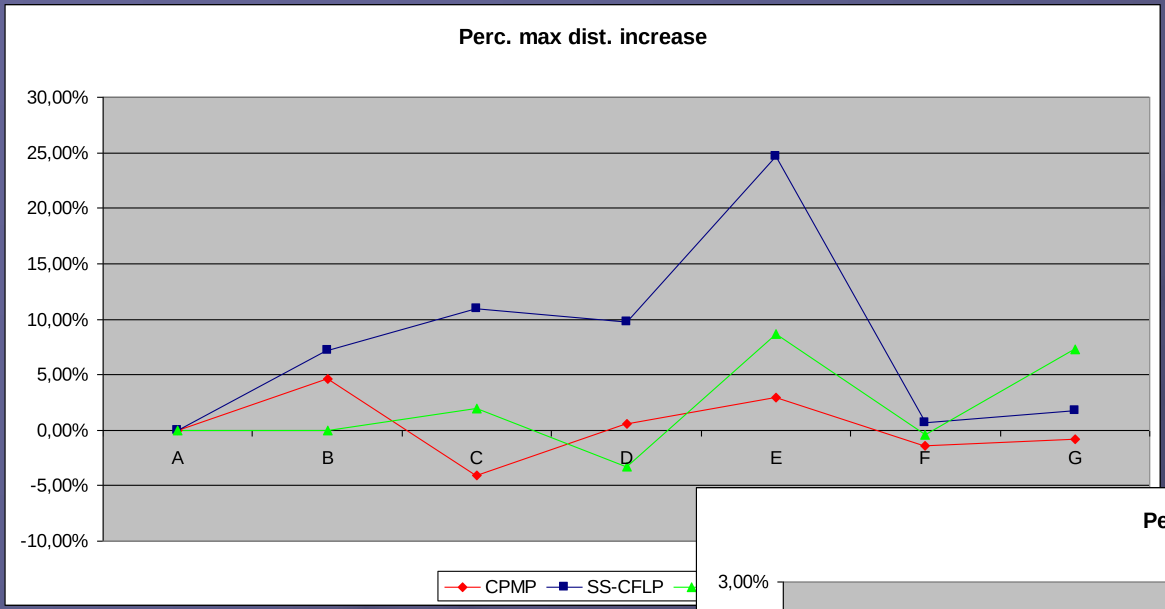


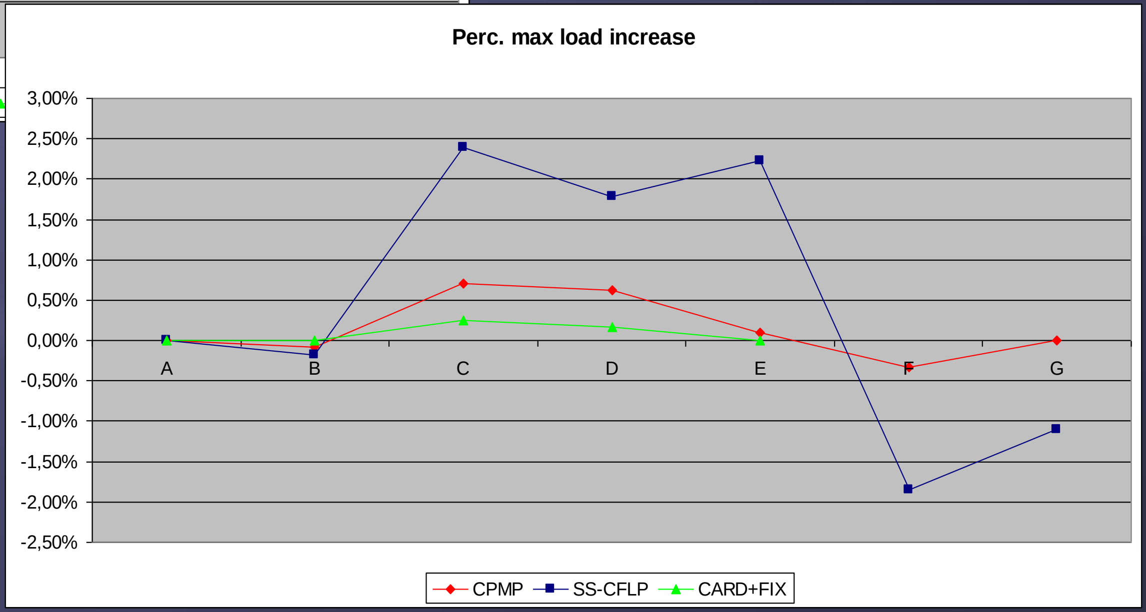
Regional constraints (optimal values)





Regional constraints (max distance and max load)





Concentrator models

		without $y_j = x_{jj}$ constraints			with $y_j = x_{jj}$ constraints		
		CPMP	SS-CFLP	CARD+FIX	CPMP	SS-CFLP	CARD+FIX
N = 50	Perc. Solved instances	100,00%	100,00%	100,00%	100,00%	100,00%	100,00%
	Avg. Computing time (s)	146,57	3,24	9,80	130,61	3,69	10,06
N = 100	Perc. Solved instances	87,14%	95,71%	88,57%	90,00%	95,71%	91,43%
	Avg. Computing time (s)	236,75	80,95	192,19	349,37	120,55	350,07

Conclusions

- Algorithms:
  - Flexible branch-and-price tool for single-source capacitated location problems
  - Allows to attack large-size instances
  - Effectiveness depends on the type of problem
- Models:
  - Non uniform setup costs make the problem harder
  - Adding cardinality constraints does not simplify the problem
  - Regional constraints help only in median-location models
  - Regional constraints have little effect on both avg. distance and max distance
  - Concentrator-like restrictions are useful only for large-size problems